An algebraic relation between consimilarity and similarity of complex matrices and its applications

2006 J. Phys. A: Math. Gen. 39 9215

(http://iopscience.iop.org/0305-4470/39/29/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 9215–9222 doi:10.1088/0305-4470/39/29/014

An algebraic relation between consimilarity and similarity of complex matrices and its applications

Tongsong Jiang[1,2], Xuehan Cheng[1,2] and Li Chen[3]

[1] Department of Mathematics, Linyi Normal University, Shandong 276005, People's Republic of China
[2] Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China
[3] Department of Physics, Linyi Normal University, Shandong 276005, People's Republic of China

E-mail: tsjemail@163.com

Received 21 March 2006, in final form 16 May 2006
Published 5 July 2006
Online at stacks.iop.org/JPhysA/39/9215

Abstract
An antilinear operator in complex vector spaces is an important operator in the study of modern quantum theory, quantum and semiclassical optics, quantum electronics and quantum chemistry. Consimilarity of complex matrices arises as a result of studying an antilinear operator referred to different bases in complex vector spaces, and the theory of consimilarity of complex matrices plays an important role in the study of quantum theory. This paper, by means of a real representation of a complex matrix, studies the relation between consimilarity and similarity of complex matrices, sets up an algebraic bridge between consimilarity and similarity and turns the theory of consimilarity into that of ordinary similarity. This paper also gives some applications of consimilarity of complex matrices.

PACS number: 02.10.Yn
Mathematics Subject Classification: 15A05, 15A21

1. Introduction

In the study of time reversal in quantum mechanics, quantum and semiclassical optics, quantum electronics and quantum chemistry [1–4], an antilinear operator in complex vector spaces plays an important role. For example, the motion of a charged particle in a given electric field is the following Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + q\phi(\mathbf{r})\right]\psi(\mathbf{r},t),$$

as we know that if $\psi(\mathbf{r},t)$ is a solution to this version of the time-dependent Schrödinger equation, then $\psi^*(\mathbf{r},-t)$($*$ is a conjugate operator) is also a solution to this version, and we take the latter as the time-reversed solution. It is necessary to take the complex conjugate because without it, the left-hand side of the equation would change sign under $t \to -t$. It is seen from this example that time reversal in quantum mechanics is represented by an antilinear operator (conjugate operator).

In general, an antilinear operator T is a mapping from one complex vector space V to another W, which is additive and conjugate homogeneous, i.e. for all $\alpha, \beta \in V$ and any complex number a,

$$T(\alpha+\beta) = T(\alpha) + T(\beta), \qquad T(a\alpha) = a^* T(\alpha), \tag{1.1}$$

where a^* is the conjugate of a. It is clear that the conjugate operator is a special antilinear operator. An antilinear operator and a linear operator are two kinds of operators in complex vector spaces.

Two $n \times n$ complex matrices A, B are said to be consimilar if $S^{-1}AS^* = B$ for some $n \times n$ nonsingular complex matrix S. Consimilarity of complex matrices arises as a result of studying an antilinear operator referred to different bases in complex vector spaces, and the theory of consimilarity of complex matrices plays an important role in the study of modern quantum theory. Consimilarity and similarity are two different equivalent relations of complex matrices.

Let $\mathbf{R}$ denote the real number field and $\mathbf{C}$ the complex number field. For $x \in \mathbf{C}$, x^* is the conjugate of x. $\mathrm{F}^{m\times n}$ denotes the set of $m \times n$ matrices on a field F. For a given matrix $A \in \mathbf{C}^{m\times n}$, A^T denotes the transpose of A, A^* the conjugate of A and A^H the conjugate transpose of A. Write $A \overset{s}{\sim} B$ if A is similar to B, $A \overset{cs}{\sim} B$ if A is consimilar to B and $A \overset{ps}{\sim} B$ if A is permutation similar to B. Permutation similarity is both similarity and consimilarity relations.

Horn and Johnson [5, chapter 4.6] and Hong and Horn [6, 7] studied the theory of consimilarity of complex matrices by means of coneigenvalues and coneigenvectors, derived a canonical form under consimilarity and gave an algebraic relation between consimilarity and ordinary similarity of complex matrices.

Lemma 1.1 ([6, 7])**.** *Let $A, B \in \mathbf{C}^{n\times n}$. Then complex matrices A and B are consimilar if and only if AA^* and BB^* are similar and A, B satisfy the alternating-product rank condition, i.e.* $\operatorname{rank} \prod^k (AA^*) = \operatorname{rank} \prod^k (BB^*)$, $k = 1, 2, \ldots, n$.

This paper, by means of a real representation of a complex matrix, studies the relation between consimilarity and ordinary similarity of complex matrices, derives a new algebraic relation theorem, sets up an algebraic bridge between consimilarity and similarity and turns the theory of consimilarity into that of ordinary similarity. This paper also gives some applications on consimilarity of complex matrices.

First of all, let us recall a lemma about a real matrix.

Lemma 1.2 ([8 chapter 6.7])**.** *Let $A \in \mathbf{R}^{n\times n}$ be a real matrix. Then,*

(1) The imaginary eigenvalues of the real matrix A appear in conjugate pairs and

(2) A is similar to a real block-diagonal matrix, each block of which has one of the two forms $J_{r_j}(\lambda_j, \lambda_j^*)$ and $J_{r_k}(\lambda_k)$, and there exists a nonsingular real matrix $T \in \mathbf{R}^{n\times n}$ such that

$$T^{-1}AT = \sum_j \oplus J_{r_j}(\lambda_j, \lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k), \tag{1.2}$$

where $\lambda_j = a_j + \mathrm{i}b_j$ are imaginary eigenvalues and λ_k are real eigenvalues of the real matrix A, and

$$J_{r_j}(\lambda_j, \lambda_j^*) = \begin{bmatrix} F_j & I_2 & & \\ & F_j & \ddots & \\ & & \ddots & I_2 \\ & & & F_j \end{bmatrix}_{r_j \times r_j}, \qquad F_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \tag{1.3}$$

and $J_{r_k}(\lambda_k)$ are the Jordan blocks of λ_k, i.e.

$$J_{r_k}(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}_{r_k \times r_k}. \tag{1.4}$$

2. Real representation

Let $A \in \mathbf{C}^{n\times n}$, and A can be uniquely written as $A = A_1 + A_2\mathrm{i}$, $A_1, A_2 \in \mathbf{R}^{n\times n}$, $\mathrm{i}^2 = -1$. The real representation matrix is defined [9] in the form

$$A^f = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{2.1}$$

the real representation matrix A^f is called the real representation of A.

Let I_n be the $n \times n$ identity matrix,

$$P_n = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \qquad Q_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Then, $P_n^2 = I_{2n}$, $Q_n^2 = -I_{2n}$.

It is easy to verify the following equations (2.2)–(2.4) by the definition of the real representation of a complex matrix. That is, if $A, B \in \mathbf{C}^{n\times n}$, then

$$(AB)^f = A^f P_n B^f = P_n (A^*)^f B^f = A^f (B^*)^f P_n, \tag{2.2}$$

$$P_n A^f P_n = (A^*)^f, \qquad Q_n A^f Q_n = A^f, \tag{2.3}$$

$$(A^T)^f = (A^f)^T, \qquad (A^H)^f = P_n (A^f)^H P_n = P_n (A^f)^T P_n, \tag{2.4}$$

and if $A \in \mathbf{C}^{n\times n}$, then A is nonsingular if and only if A^f is nonsingular.

For $A \in \mathbf{C}^{n\times n}$ and $\alpha \in \mathbf{C}^{2n\times 1}$, if $A^f\alpha = \lambda\alpha$, then by (2.3) we have

$$A^f \alpha^* = \lambda^* \alpha^*, \qquad A^f(Q_n\alpha) = -\lambda(Q_n\alpha), \qquad A^f(Q_n\alpha^*) = -\lambda^*(Q_n\alpha^*). \tag{2.5}$$

In the same manner, for a Jordan block $J_r(\lambda)$ of an eigenvalue λ, if $A^f P = P J_r(\lambda)$, then by (2.3) we have

$$A^f P^* = P^* J_r(\lambda^*), \qquad A^f(Q_n P) = -(Q_n P) J_r(\lambda), \qquad A^f(Q_n P^*) = -(Q_n P^*) J_r(\lambda^*). \tag{2.6}$$

From the above statement, we have the following result.

Proposition 2.1. *Let $A \in \mathbf{C}^{n\times n}$. Then,*

(1) the eigenvalues of the real representation A^f appear in positive and negative conjugate pairs, i.e. if λ is an imaginary eigenvalue of A^f, then $\pm\lambda, \pm\lambda^*$ are also imaginary eigenvalues of A^f, and if λ is a real eigenvalue of A^f, then $-\lambda$ is also a real eigenvalue of A^f;

(2) the Jordan blocks $J_r(\lambda)$ of the real representation A^f appear in positive and negative conjugate pairs, i.e. if $J_r(\lambda)$ is a Jordan block of an imaginary eigenvalue of A^f, then $\pm J_r(\lambda), \pm J_r(\lambda^*)$ are also Jordan blocks, and if $J_r(\lambda)$ is a Jordan block of a real eigenvalue of A^f, then $-J_r(\lambda)$ is also a real Jordan block of A^f.

3. An algebraic relation between consimilarity and similarity

This section gives an algebraic relation between consimilarity and similarity of complex matrices by means of a real representation of a complex matrix.

Let $A \in \mathbf{C}^{n\times n}$. Clearly, by direct calculation we get that $J_{r_k}(-\lambda_k)$ is similar to $-J_{r_k}(\lambda_k)$ and $J_{r_j}(-\lambda_j, -\lambda_j^*)$ is similar to $-J_{r_j}(\lambda_j, \lambda_j^*)$. Therefore, by lemma 1.2 and proposition 2.1, we have

$$A^f \overset{s}{\sim} \sum_j \oplus \begin{bmatrix} J_{r_j}(\lambda_j,\lambda_j^*) & 0 \\ 0 & J_{r_j}(-\lambda_j,-\lambda_j^*) \end{bmatrix} \oplus \sum_k \oplus \begin{bmatrix} J_{r_k}(\lambda_k) & 0 \\ 0 & J_{r_k}(-\lambda_k) \end{bmatrix} \tag{3.1}$$

$$\overset{s}{\sim} \sum_j \oplus \begin{bmatrix} J_{r_j}(\lambda_j,\lambda_j^*) & 0 \\ 0 & -J_{r_j}(\lambda_j,\lambda_j^*) \end{bmatrix} \oplus \sum_k \oplus \begin{bmatrix} J_{r_k}(\lambda_k) & 0 \\ 0 & -J_{r_k}(\lambda_k) \end{bmatrix} \tag{3.2}$$

$$\overset{ps}{\sim} \begin{bmatrix} \sum_j \oplus J_{r_j}(\lambda_j,\lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k) & 0 \\ 0 & -\sum_j \oplus J_{r_j}(\lambda_j,\lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k) \end{bmatrix} \tag{3.3}$$

$$= \left(\sum_j \oplus J_{r_j}(\lambda_j,\lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k) \right)^f, \tag{3.4}$$

where $\lambda_j = a_j + \mathrm{i}b_j$ $(a_j \geqslant 0, b_j > 0)$ are imaginary eigenvalues and $\lambda_k(\geqslant 0)$ are real eigenvalues of the real representation A^f, $J_{r_j}(\lambda_j, \lambda_j^*)$ and $J_{r_k}(\lambda_k)$ are the forms of the real matrices given in (1.3) and (1.4).

From lemma 1.2 and the above statement, we get the following result.

Proposition 3.1. *Let $A \in \mathbf{C}^{n\times n}$. Then there exists a nonsingular real matrix $T \in \mathbf{R}^{n\times n}$ such that*

$$T^{-1}A^fT = \left(\sum_j \oplus J_{r_j}(\lambda_j,\lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k) \right)^f, \tag{3.5}$$

where $\lambda_j = a_j + \mathrm{i}b_j$ $(a_j \geqslant 0, b_j > 0)$ are imaginary eigenvalues and $\lambda_k(\geqslant 0)$ are real eigenvalues of the real representation A^f, and $J_{r_j}(\lambda_j, \lambda_j^*)$ and $J_{r_k}(\lambda_k)$ are the forms of the real matrices given in (1.3) and (1.4).

Let $A, B \in \mathbf{C}^{n\times n}$ be two complex matrices. If A is consimilar to B, then there exists a nonsingular complex matrix S such that $AS^* = SB$, by (2.2) $A^fS^fP_n = S^fP_nB^f$. This means that if A is consimilar to B, then A^f is similar to B^f.

Conversely, if A^f is similar to B^f, then A^f and B^f have the same eigenvalues; by (3.2) and [8, chapter 6.7] there exists a real and full-rank matrix X_j such that $A^f X_j = X_j J_r(\lambda_j, \lambda_j^*)$. Then by $Q_n A^f Q_n = A^f$ in (2.3), we have

$$A^f X_j = X_j J_{r_j}(\lambda_j, \lambda_j^*), \qquad A^f(Q_n X) = -(Q_n X) J_{r_j}(\lambda_j, \lambda_j^*), \tag{3.6}$$

and (3.6) is equivalent to

$$A^f(X_j, Q_n X_j) = (X_j, Q_n X_j)\begin{bmatrix} J_{r_j}(\lambda_j, \lambda_j^*) & 0 \\ 0 & -J_{r_j}(\lambda_j, \lambda_j^*) \end{bmatrix}. \tag{3.7}$$

In a similar manner, there exists a real and full-rank matrix Y_k such that

$$A^f(Y_k, Q_n Y_k) = (Y_k, Q_n Y_k)\begin{bmatrix} J_{r_k}(\lambda_k) & 0 \\ 0 & -J_{r_k}(\lambda_k) \end{bmatrix}. \tag{3.8}$$

Combining (3.2)–(3.8) there exists a nonsingular real matrix $T = (Z, Q_n Z) \in \mathbf{R}^{2n\times 2n}$, $Z \in \mathbf{R}^{2n\times n}$ such that

$$A^f T = T J^f, \qquad J = \sum_j \oplus J_{r_j}(\lambda_j, \lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k). \tag{3.9}$$

Let $Z = \left[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right]$, $Z_1, Z_2 \in \mathbf{R}^{n\times n}$ and $S = Z_1 + Z_2\mathrm{i}$. Then, $T = S^f P_n$. From the nonsingular matrix T, we get S which is a nonsingular matrix, and (3.9) is equivalent to

$$A^f S^f P_n = S^f P_n J^f \quad \Leftrightarrow \quad (AS^*)^f = (SJ)^f \quad \Leftrightarrow \quad AS^* = SJ. \tag{3.10}$$

This means that A is consimilar to J. Similarly, B is consimilar to J. Therefore, A is consimilar to B.

The above statement implies the following result.

Theorem 3.2. *Let $A, B \in \mathbf{C}^{n\times n}$. Then A is consimilar to B if and only if A^f is similar to B^f, i.e. $A \overset{cs}{\sim} B$ if and only if $A^f \overset{s}{\sim} B^f$.*

Clearly for any $A \in \mathbf{C}^{n\times n}$, A^f is similar to $(A^f)^T$, and it is clear by (2.3) and (2.4) that

$$A^f \overset{s}{\sim} (A^*)^f, \qquad A^f \overset{s}{\sim} (A^T)^f, \qquad A^f \overset{s}{\sim} (A^H)^f.$$

Combining the above statement and theorem 3.2, we have the following result.

Corollary 3.3. *Let $A \in \boldsymbol{C}^{n\times n}$. Then $A \overset{cs}{\sim} A^*$, $A \overset{cs}{\sim} A^T$ and $\overset{cs}{\sim} A^H$.*

Theorem 3.2 gives an algebraic relation between consimilarity and similarity of complex matrices by means of a real representation, sets up a bridge between consimilarity and similarity and turns the theory of consimilarity into that of similarity by means of a real representation of a complex matrix.

4. Applications

By proposition 3.1 and theorem 3.2, we get a real concanonical form of a complex matrix under consimilarity.

Theorem 4.1. *Every $n \times n$ complex matrix A is consimilar to a real concanonical form, and the real concanonical form is unique up to permutation of the diagonal canonical blocks. Specifically, there exists a nonsingular $n \times n$ complex matrix S, such that*

$$S^{-1}AS^* = \sum_j \oplus J_{r_j}(\lambda_j, \lambda_j^*) \oplus \sum_k \oplus J_{r_k}(\lambda_k), \tag{4.1}$$

where

$$J_{r_j}(\lambda_j, \lambda_j^*) = \begin{bmatrix} F_j & I_2 & & \\ & F_j & \ddots & \\ & & \ddots & I_2 \\ & & & F_j \end{bmatrix}, \qquad F_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \tag{4.2}$$

where $\lambda_j = a_j + \mathrm{i}b_j$ $(a_j \geqslant 0, b_j > 0)$, $\lambda_k \geqslant 0$ *are the eigenvalues of* A^f *and* $J_{r_k}(\lambda_k)$ *are the Jordan blocks of* $\lambda_k \geqslant 0$.

The following result immediately comes from theorem 4.1

Corollary 4.2. *Every* $n \times n$ *complex matrix is consimilar to an* $n \times n$ *real matrix.*

Combining lemma 1.1, theorems 3.2 and 4.1, we get the following result.

Corollary 4.3. *Let* $A, B \in \mathbf{C}^{n\times n}$. *Then the following statements are equivalent:*

(1) A *and* B *are consimilar;*
(2) the real representation of A *and* B *is similar;*
(3) AA^* *and* BB^* *are similar and* A, B *satisfy the alternating-product rank condition, i.e.* $\operatorname{rank} \prod^k (AA^*) = \operatorname{rank} \prod^k (BB^*)$, $k = 1, 2, \ldots, n$ *and*
(4) A *and* B *have the same concanonical form.*

5. Example

This section, by means of a real representation of a complex matrix, gives possible methods for finding a nonsingular complex matrix S with $S^{-1}AS^* = J$.

First of all, by proposition 3.1 if there exists a nonsingular real matrix T such that $A^f T = T J^f$ and by (2.3) we have $Q_n A^f Q_n = A^f$ and $Q_n J^f Q_n = J^f$, then

$$A^f T = T J^f \quad \Leftrightarrow \quad A^f(Q_n T Q_n) = (Q_n T Q_n) J^f. \tag{5.1}$$

Let

$$\hat{T}_1 = \tfrac{1}{2}(T - Q_n T Q_n), \qquad \hat{T}_2 = \tfrac{1}{2}(T + Q_n T Q_n). \tag{5.2}$$

Then

$$A^f \hat{T}_l = \hat{T}_l J^f, \qquad l = 1, 2. \tag{5.3}$$

Let

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{5.4}$$

where $T_{kl} \in \mathbf{R}^{n\times n}$. It is easy to get by direct calculation

$$\hat{T}_1 = \begin{bmatrix} \hat{T}_{11} & -\hat{T}_{12} \\ \hat{T}_{12} & \hat{T}_{11} \end{bmatrix}, \qquad \hat{T}_2 = \begin{bmatrix} \hat{T}_{21} & \hat{T}_{22} \\ \hat{T}_{22} & -\hat{T}_{21} \end{bmatrix}, \tag{5.5}$$

where

$$\hat{T}_{11} = \tfrac{1}{2}(T_{11} + T_{22}), \qquad \hat{T}_{12} = \tfrac{1}{2}(T_{21} - T_{12}), \tag{5.6}$$

$$\hat{T}_{21} = \tfrac{1}{2}(T_{11} - T_{22}), \qquad \hat{T}_{22} = \tfrac{1}{2}(T_{21} + T_{12}). \tag{5.7}$$

From (5.5), we construct two complex matrices

$$S_1 = \hat{T}_{11} + \hat{T}_{12}\mathrm{i} = \frac{1}{2}(I_n, \mathrm{i}I_n)\hat{T}_1 \begin{bmatrix} I_n \\ -\mathrm{i}I_n \end{bmatrix} = \frac{1}{4}(I_n, \mathrm{i}I_n)(T - Q_n T Q_n) \begin{bmatrix} I_n \\ -\mathrm{i}I_n \end{bmatrix}, \tag{5.8}$$

$$S_2 = \hat{T}_{21} + \hat{T}_{22}\mathrm{i} = \frac{1}{2}(I_n, \mathrm{i}I_n)\hat{T}_2 \begin{bmatrix} I_n \\ \mathrm{i}I_n \end{bmatrix} = \frac{1}{4}(I_n, \mathrm{i}I_n)(T + Q_n T Q_n) \begin{bmatrix} I_n \\ \mathrm{i}I_n \end{bmatrix}. \tag{5.9}$$

Clearly, $S_1^f P_n = \hat{T}_1$ and $S_2^f = \hat{T}_2$. Then by (2.2), equation (5.3) is equivalent to

$$A^f S_1^f P_n = S_1^f P_n J^f \quad \Leftrightarrow \quad (AS_1^*)^f = (S_1 J)^f \quad \Leftrightarrow \quad AS_1^* = S_1 J, \tag{5.10}$$

and

$$A^f S_2^f P_n = S_2 J^f P_n \quad \Leftrightarrow \quad (AS_2^*)^f = (S_2 J^*)^f = (S_2 J)^f \quad \Leftrightarrow \quad AS_2^* = S_2 J. \tag{5.11}$$

If either of S_1 and S_2 is nonsingular, then we find a nonsingular complex matrix S with $S^{-1}AS^* = J$. Therefore, the above statement provides two possible methods for finding a nonsingular complex matrix S with $S^{-1}AS^* = J$.

Example. Let A be a complex matrix

$$A = \begin{bmatrix} \mathrm{i} & 1+\mathrm{i} & 1-\mathrm{i} \\ -2+\mathrm{i} & 0 & 1+2\mathrm{i} \\ -1 & -\mathrm{i} & 1+\mathrm{i} \end{bmatrix}.$$

Find the real concanonical form J and a nonsingular complex matrix S such that $S^{-1}AS^* = J$.

Solution. It is easy by (2.1) that the real representation A^f of the complex matrix A is

$$A^f = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 \\ -2 & 0 & 1 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{bmatrix}$$

and the eigenvalues of A^f are $\lambda_1 = 1+2\mathrm{i}$, $\lambda_2 = 1-2\mathrm{i} = \lambda_1^*$, $\lambda_3 = -1-2\mathrm{i} = -\lambda_1$, $\lambda_4 = -1+2\mathrm{i} = -\lambda_1^*$, $\lambda_5 = 1$ and $\lambda_6 = -1 = -\lambda_5$. From (1.3), let

$$J_1(\lambda_1, \lambda_1^*) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \qquad J_1(\lambda_5) = (1).$$

By direct calculation, we can find a real nonsingular matrix T (3.5) such that

$$T^{-1}A^f T = (J_1(\lambda_1, \lambda_1^*) \oplus J_1(\lambda_5))^f = J^f = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^f,$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, from (5.8) we let

$$S = \frac{1}{4}(I_3, \mathrm{i}I_3)(T - Q_3TQ_3)\begin{bmatrix} I_3 \\ -\mathrm{i}I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{i} \\ \mathrm{i} & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and clearly S is a nonsingular complex matrix, so by theorems 3.2 and 4.1, the real matrix J is the concanonical form of the complex matrix A under consimilarity and S is a nonsingular complex matrix such that

$$S^{-1}AS^* = J = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Acknowledgments

This paper is supported by the National Natural Science Foundation of China and Shandong Natural Science Foundation of China (Y2005A12).

References

[1] Sakurai J J 1985 *Modern Quantum Mechanics* (Menlo Park, CA: Benjamin-Cummings)
[2] Weinberg S 1995 *The Quantum Theory of Fields* vol 1 (Cambridge: Cambridge University Press)
[3] Kurucz Z, Koniorczyk M Adam P and Janszky J 2003 An operator description of entanglement matching in quantum teleportation *J. Opt. B: Quantum Semiclass. Opt.* **5** 627–32
[4] Levine I N 1991 *Quantum Chemistry* 4th edn (Englewood Cliffs, NJ: Prentice-Hall)
[5] Horn R A and Johnson C R 1985 *Matrix Analysis* (New York: Cambridge University Press)
[6] Hong Y P 1985 Consimilarity: theory and applications *Doctoral dissertation* Johns Hopkins University
[7] Hong Y P and Horn R A 1988 A canonical form for matrices under consimilarity *Linear Algebra Appl.* **102** 143–68
[8] Lancaster P and Tismenetsky M 1985 *The Theory of Matrices with Applications* 2nd edn (New York: Academic)
[9] Jiang T and Wei M 2003 On solutions of the matrix equations $X - AXB = C$ and $X - A\overline{X}B = C$ *Linear Algebra Appl.* **367** 225–33